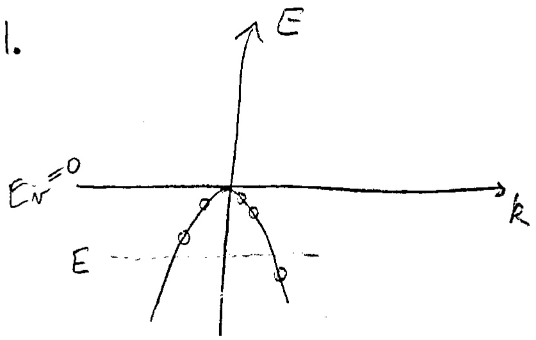
ECE 6237 Optical Detectors HW2 Solution

1.



$$p = \int_{-\infty}^{E_v} (1-f(E))\, N(E)\, dE$$

Band energy with $E_v = 0$

$$E = -\frac{\hbar^2 k^2}{2m^*}$$

$N_T(E)$ = # of valence band states with $E' > E$

$$= \frac{2}{(2\pi)^3} \frac{4\pi}{3} k^3(E) \quad \text{where } k(E) = \sqrt{-\frac{2m^*E}{\hbar^2}}$$

spin

k-space volume of one state

$$\Rightarrow N_T(E) = \frac{2}{(2\pi)^3} \frac{4\pi}{3} \left(-\frac{2m^*E}{\hbar^2}\right)^{3/2}$$

$$N(E) = \text{Density of states} = \frac{dN_T(E)}{dE} = \frac{2}{(2\pi)^3} \frac{4\pi}{3} \left(\frac{2m^*}{\hbar^2}\right)^{3/2} \frac{3}{2} (-E)^{1/2}$$

$$= \frac{2}{(2\pi)^2} \left(\frac{2m^*}{(h/2\pi)^2}\right)^{3/2} (-E)^{1/2}$$

$$1 - f(E) = 1 - \frac{1}{e^{(E-E_f)/kT}+1}$$

$$= \frac{e^{(E-E_f)/kT}}{e^{(E-E_f)/kT}+1} = \frac{1}{e^{-(E-E_f)/kT}+1}$$

Since for the entire valence band $E<0$ and $-(E-E_f) \gg kT$

can neglect the 1 in the denominator.

$$\Rightarrow p \simeq \int_{-\infty}^{0} \frac{1}{e^{-(E-E_f)/kT}} \frac{2}{(2\pi)^2} \left(\frac{2m^*}{(h/2\pi)^2}\right)^{3/2} (-E)^{1/2}\, dE$$

Changing integration variable from $E \to -E$ and reversing limits of integration:

$$p = \int_0^{\infty} e^{-(E+E_f)/kT} \frac{2}{(2\pi)^2} \left(\frac{2m^*}{(h/2\pi)^2}\right)^{3/2} E^{1/2}\, dE$$

$$= e^{-E_f/kT} \frac{2}{(2\pi)^2} \left(\frac{2m^*}{(h/2\pi)^2}\right)^{3/2} \int_0^{\infty} e^{-E/kT} E^{1/2}\, dE$$

Definite integral is given by: $\int_0^{\infty} x^{1/2} e^{-ax}\, dx = \frac{\sqrt{\pi}}{2a\sqrt{a}}$

$$\Rightarrow p = e^{-E_f/kT} \frac{2}{(2\pi)^2} \left(\frac{2m^*}{(h/2\pi)^2}\right)^{3/2} \frac{\sqrt{\pi}\,(kT)^{3/2}}{2}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \left(\frac{2m^* kT}{(h/2\pi)^2}\right)^{3/2} e^{-E_f/kT}$$

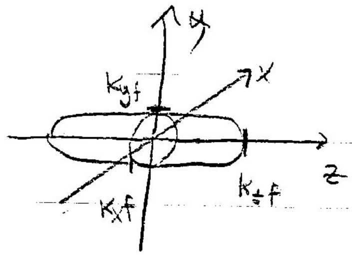
$$= 2\left(\frac{2\pi m^* kT}{h^2}\right)^{3/2} e^{-E_f/kT}$$

If we change the energy scale such that $E_v \neq 0$, have to express all energies as $E - E_v$:

$$p = 2\left(\frac{2\pi m^* kT}{h^2}\right)^{3/2} e^{-(E_f-E_v)/kT}$$

2. $$E = \frac{(\hbar k_z)^2}{2m_\ell} + \frac{(\hbar k_x)^2}{2m_t} + \frac{(\hbar k_y)^2}{2m_t}$$

Constant energy surface is ellipsoid in k-space



$$k_{yf} = k_{xf} = \sqrt{\frac{2m_t E}{\hbar^2}}$$

$$k_{zf} = \sqrt{\frac{2m_\ell E}{\hbar^2}}$$

$$\text{Volume} = \frac{4\pi}{3} k_{xf} k_{yf} k_{zf} = \frac{4\pi}{3}\left(\frac{2E}{\hbar^2}\right)^{3/2}\left(m_t^2 m_\ell\right)^{1/2}$$

$N_T(E)$ = # of states with $E' < E$

$$= \underbrace{2}_{\text{spin states}} \cdot \underbrace{6}_{\text{number of conduction band pockets}} \cdot \underbrace{\frac{1}{(2\pi)^3}}_{\text{k-space volume per electron state}} \frac{4\pi}{3}\left(\frac{2E}{\hbar^2}\right)^{3/2}\left(m_t^2 m_\ell\right)^{1/2}$$

$$N(E) = \frac{dN_T(E)}{dE} = \frac{2}{(2\pi)^3}\frac{4\pi}{3}\left[6\left(m_t^2 m_\ell\right)^{1/2}\right]\left(\frac{2}{\hbar^2}\right)^{3/2}\frac{3}{2}E^{1/2}$$

Comparing with Problem 1, we see that:

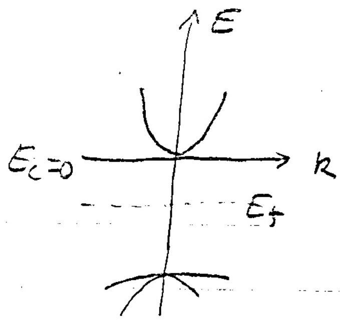
$$6\left(m_t^2 m_\ell\right)^{1/2} = \left(m_d^*\right)^{3/2}$$

$$\Rightarrow m_d = 6^{2/3}\left(0.98 \cdot 0.19 \cdot 0.19\right)^{1/3} m_0$$

$$= 1.08\ m_0$$

The rest of the derivation follows as in Problem 1

Setting $E_c = 0$



$$n = \int_0^\infty \frac{1}{e^{(E-E_f)/kT}+1} \frac{2}{(2\pi)^2}\left(\frac{2m_d^*}{\hbar^2}\right)^{3/2} E^{1/2}\, dE$$

For $E - E_f >> kT$: $\frac{1}{e^{(E-E_f)/kT}+1} \simeq \frac{1}{e^{(E-E_f)/kT}} = e^{-(E-E_f)/kT}$

$$n = e^{E_f/kT} \frac{2}{(2\pi)^2}\left(\frac{2m_d^*}{\hbar^2}\right)^{3/2} \int_0^\infty E^{1/2} e^{-E/kT}\, dE$$

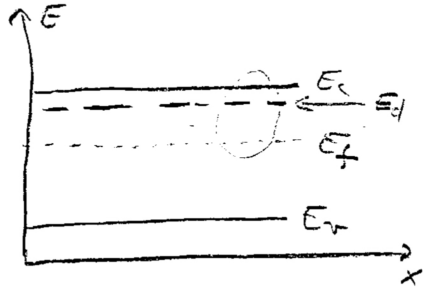
$$= e^{E_f/kT} \frac{2}{(2\pi)^2}\left(\frac{2m_d}{\hbar^2}\right)^{3/2} \frac{\sqrt{\pi}\,(kT)^{3/2}}{2}$$

$$\Rightarrow \quad n = 2\left[\frac{2\pi m_d^* kT}{h^2}\right]^{3/2} e^{E_f/kT}$$

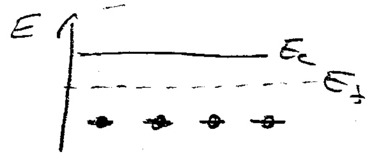
If we let $E_c \neq 0$, we have to set $E \to E - E_c$

$$\Rightarrow \quad n = 2\left[\frac{2\pi m_d^* kT}{h^2}\right]^{3/2} e^{-(E_c-E_f)/kT}$$

where $m_d^* = 1.08\, m_0$

3. 

At $T=0$, fermi level is between donor level and conduction band edge:



Set $E_v = 0$

$$\Rightarrow E_c = E_g = 1.11\,eV$$

$$E_d = E_c - 0.044\,eV = 1.066\,eV$$

Charge balance

$\Rightarrow$ # of conduction-band electrons

= # of unoccupied donor sites + # of holes

$$n = N_d^+ + p$$

$$= N_d(1 - f(E_d)) + p$$

$$n = 2\left(\frac{2\pi m_n^* kT}{h^2}\right)^{3/2} e^{-(E_c - E_f)/kT}$$

$$= (1.1)^{3/2} (T)^{3/2} (4.818 \times 10^{21})\, e^{-(E_c - E_f)/kT}$$

$$p = 2\left(\frac{2\pi m_p^* kT}{h^2}\right)^{3/2} e^{-(E_f - E_v)/kT}$$

$$= (0.55)^{3/2} (T)^{3/2} (4.818 \times 10^{21})\, e^{-(E_f - E_v)/kT}$$

$$N_d^+ = N_d\left(1 - \frac{1}{e^{(E_d - E_f)/kT} + 1}\right) = N_d\left[\frac{1}{e^{-(E_d - E_f)/kT} + 1}\right]$$

$$= 3 \times 10^{23} \frac{1}{e^{-(E_d - E_f)/kT} + 1}$$

Charge balance $\Rightarrow$

$$(1.1)^{3/2} (T)^{3/2} (4.818 \times 10^{21})\, e^{-(E_c - E_f)/kT}$$
$$= (0.55)^{3/2} (T)^{3/2} (4.818 \times 10^{21})\, e^{-(E_f - E_v)/kT}$$
$$+ 3 \times 10^{23} \left[e^{-(E_d - E_f)/kT} + 1 \right]^{-1}$$

Divide by 10^{21} and put in values for E_c, E_d, and E_v ($E_v = 0$)

$$(4.818)(T^{3/2})\left[(1.1)^{3/2} e^{-(1.11 - E_f)/kT} - (0.55)^{3/2} e^{-E_f/kT}\right]$$
$$- 300 \left[e^{-(1.066 - E_f)/kT} + 1 \right]^{-1} = 0$$

Define the left-hand-side of equation above as $f(E_f)$.
Numerically solve to see what value of E_f makes $f(E_f) = 0$.

C:\MATLAB6p5\work\fermi_level.m
March 16, 2004

```
function y=fermi_level(Ef)
T=4.2;
kT=T*1.38e-23/1.6e-19;
exp1=exp( -(1.11-Ef)/kT );
exp2=exp( -Ef/kT );
exp3=exp( -(1.066-Ef)/kT );
y=4.818*T^1.5 *(1.1^1.5 *exp1 - 0.55^1.5 *exp2) - 300 ./ (exp3 +1);
```

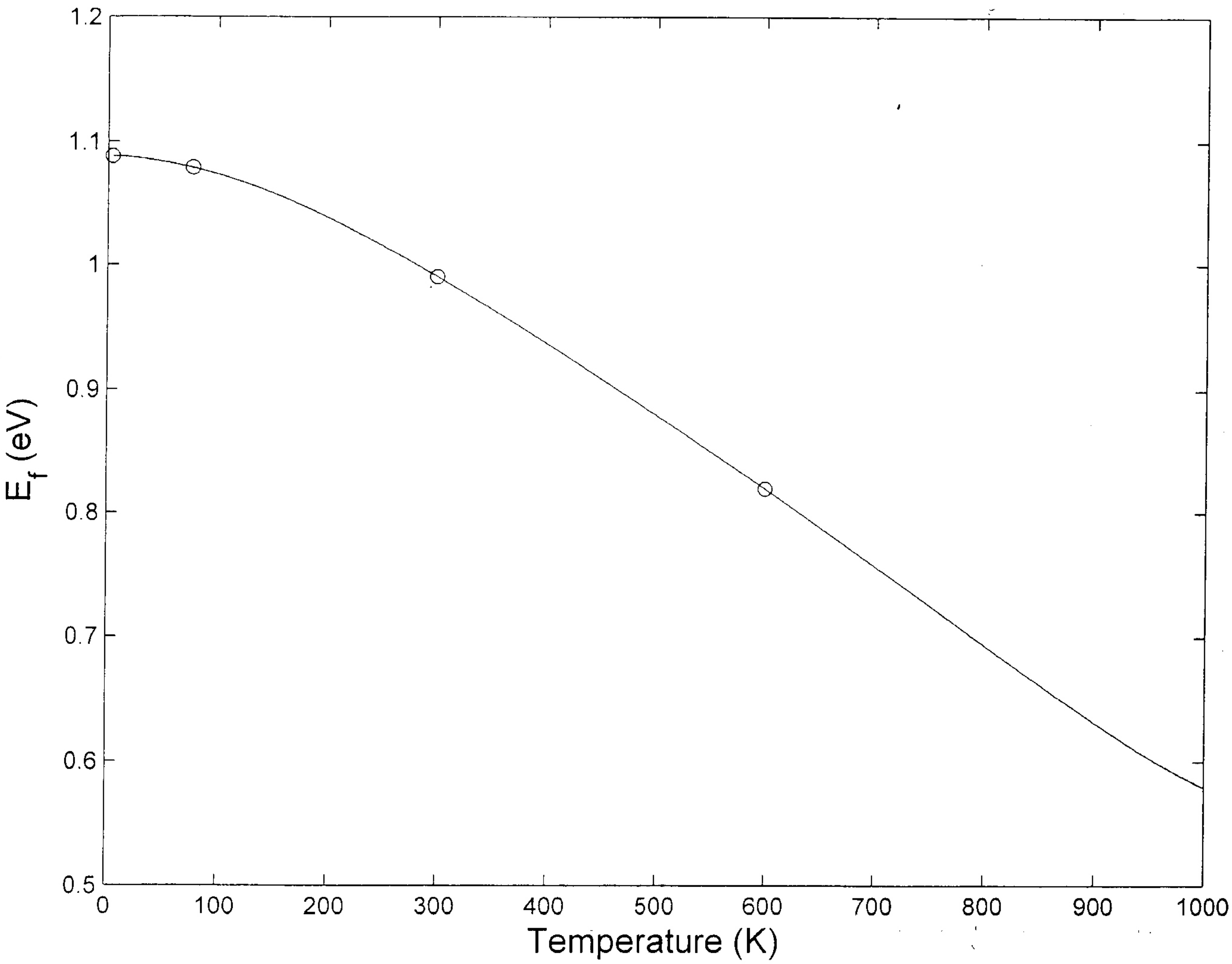
a) $T = 4.2K \Rightarrow kT = 0.000362$ eV
From MATLAB, $E_f = 1.0833$
$E_c - E_f = 0.0217$ eV $\quad (E_c - E_f)/kT = 59.8$

b) $T = 77K \Rightarrow kT = 0.00664$ eV
From MATLAB: $E_f = 1.0799$; $E_c - E_f = 0.0308 = 4.64\ kT$

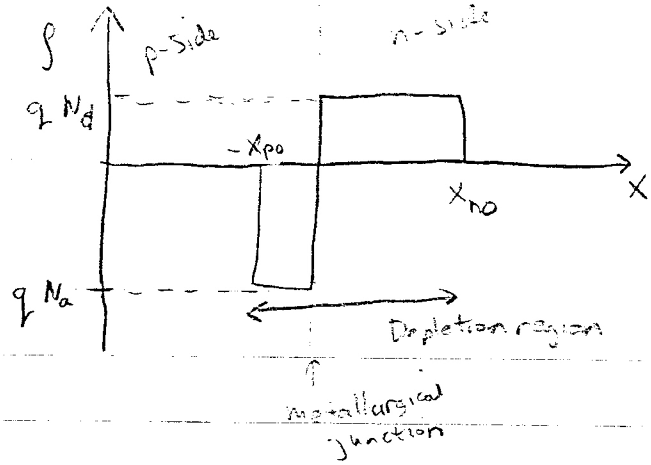
c) $T = 300K \Rightarrow kT = 0.0259$ eV
From MATLAB: $E_f = 0.9905 \quad (E_c - E_f)/kT = 4.62$

d) $T = 600K \Rightarrow kT = 0.0517$
From MATLAB: $E_f = 0.8194 \quad (E_c - E_f)/kT = 5.62$

```
>> global T
>> T=4.2
T =
    4.2000
>> Ef42=fzero('fermi_level', 1.0)
Ef42 =
    1.0883
>> T=77
T =
    77
>> Ef77=fzero('fermi_level', 1.0)
Ef77 =
    1.0792
>> T=300
T =
   300
>> Ef300=fzero('fermi_level', 1.0)
Ef300 =
    0.9905
>> T=600
T =
   600
>> Ef600=fzero('fermi_level', 1.0)
Ef600 =
    0.8194
>>
```



4. Abrupt junction:



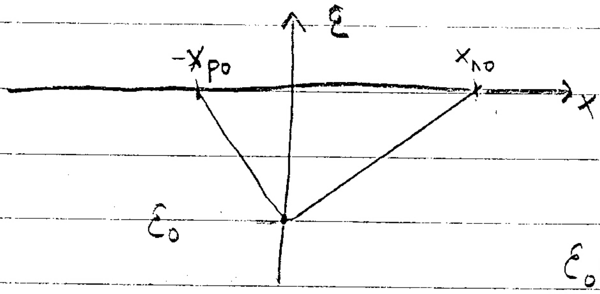
Stored charge $= Q_0 = A q N_a x_{op} = A q N_d x_{on}$

Relation of junction width to applied voltage:

Poisson's equation: $\nabla \cdot \epsilon \mathcal{E} = \rho$

$$\epsilon \frac{d\mathcal{E}}{dx} = \rho$$

Integrating across junction:



$$\mathcal{E}_0 = -\frac{q}{\epsilon} N_a x_{po} = -\frac{q}{\epsilon} N_d x_{no}$$

For $x<0$:

$$\mathcal{E}(x) = \int_{-x_{po}}^{x} -\frac{q}{\epsilon} N_a \, dx = -\frac{q}{\epsilon} N_a (x + x_{po})$$

For $x>0$:

$$\mathcal{E}(x) = \int_{x_{no}}^{x} \frac{q}{\epsilon} N_d \, dx = \frac{q}{\epsilon} N_d (x - x_{no})$$

$$\mathcal{E}(x) = -\frac{dV(x)}{dx}$$

$$\Rightarrow \Delta V = -\int \mathcal{E}(x)\,dx$$

Integrating across entire junction yields: $\Delta V = -\frac{1}{2}\mathcal{E}_0 (x_{p0} + x_{n0})$

$$= \frac{q N_a}{2\epsilon} x_{p0}(x_{p0} + x_{n0})$$

Under bias:

$\Delta V \rightarrow V_0 - V_a$

$x_{p0} \rightarrow x_p$

$x_{n0} \rightarrow x_n$

$$V_0 - V_a = \frac{q}{2}\frac{N_a}{\epsilon} x_p (x_p + x_n)$$

Stored charge $= Q = A q N_a x_p = A q N_d x_n$

$$\Rightarrow x_n = \frac{N_a}{N_d} x_p$$

$$V_0 - V_a = \frac{q}{2}\frac{N_a}{\epsilon} x_p \left(x_p + \frac{N_a}{N_d} x_p\right) = \frac{q}{2}\frac{N_a}{\epsilon} x_p^2 \left(1 + \frac{N_a}{N_d}\right)$$

$$2\epsilon (V_0 - V_a)\left(1 + \frac{N_a}{N_d}\right)^{-1} = \frac{1}{q N_a A^2} Q^2$$

$$Q = \sqrt{2\epsilon q N_a A^2 (V_0 - V_a)\left(1 + \frac{N_a}{N_d}\right)^{-1}}$$

$$C_j = \frac{dQ}{dV_a} = \frac{dQ}{d(V_0 - V_a)}\frac{d(V_0 - V_a)}{dV_a} = -\frac{dQ}{d(V_0 - V_a)} = \frac{1}{2}\frac{1}{Q}\left(2\epsilon q N_a A^2 \left(1 + \frac{N_a}{N_d}\right)^{-1}\right)$$

$$C_j = \frac{1}{2}\frac{2\epsilon q N_a A^2 \left(1 + \frac{N_a}{N_d}\right)^{-1}}{\sqrt{2\epsilon q N_a A^2 (V_0 - V_a)\left(1 - N_a/N_d\right)^{-1}}} = \frac{A}{2}\sqrt{\frac{2q\epsilon N_a \left(1 + N_a/N_d\right)^{-1}}{(V_0 - V_a)}}$$

$$C_j = \frac{A}{2}\sqrt{\frac{2q\epsilon}{(V_0 - V_a)}\frac{N_a N_d}{(N_a + N_d)}}$$

Approximate parallel-plate capacitor solution:

$$C' = \epsilon \frac{A}{d} = \epsilon \frac{A}{W}$$

$$\Delta V = (V_0 - V_a) = \frac{q N_a}{2\epsilon} x_p (x_p + x_n)$$

$$(x_p + x_n) = W$$

$$x_p \left(1 + \frac{N_a}{N_d}\right) = W \Rightarrow x_p = W \left(1 + \frac{N_a}{N_d}\right)^{-1}$$

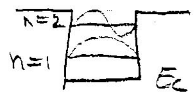
$$\Rightarrow (V_0 - V_a) = \frac{q N_a}{2\epsilon} W^2 \left(1 + \frac{N_a}{N_d}\right)^{-1}$$

$$W = \left[\frac{2\epsilon (V_0 - V_a)}{q \frac{N_a N_d}{(N_a + N_d)}}\right]^{1/2}$$

$$C' = \frac{\epsilon A}{\sqrt{\frac{2\epsilon (V_0 - V_a)}{q \frac{N_a N_d}{(N_a + N_d)}}}} = A \sqrt{\frac{\epsilon q}{2(V_0 - V_a)} \frac{N_a N_d}{(N_a + N_d)}}$$

This is the same as the exact result!

5.



$$E = E_c + E_n + \frac{(\hbar k_x)^2}{2m^*} + \frac{(\hbar k_y)^2}{2m^*}$$

where $E_n = \frac{\hbar^2}{2m^*}(n\pi/d)^2 \qquad n = 1, 2, 3 \ldots$

Setting $E_c = 0$:

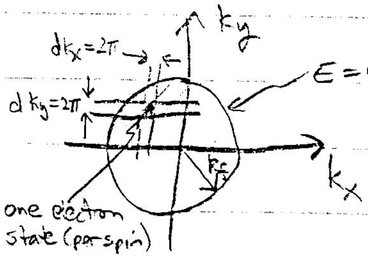
$$E = E_n + \frac{\hbar^2}{2m^*}(k_x^2 + k_y^2) = \frac{\hbar^2}{2m^*}\left[\left(\frac{n\pi}{d}\right)^2 + k_t^2\right]$$

(See plot)

Density of states $(n=1)$

$$N_1(E) = 0 \quad \text{for} \quad E < E_1 = \frac{\hbar^2}{2m^*}\left(\frac{\pi}{d}\right)^2$$

For $E > E_1$, need to count states in k-space



$$E = E' = \frac{\hbar^2}{2m^*}(k_x^2 + k_y^2) + \frac{\hbar^2}{2m^*}\left(\frac{n\pi}{d}\right)^2$$

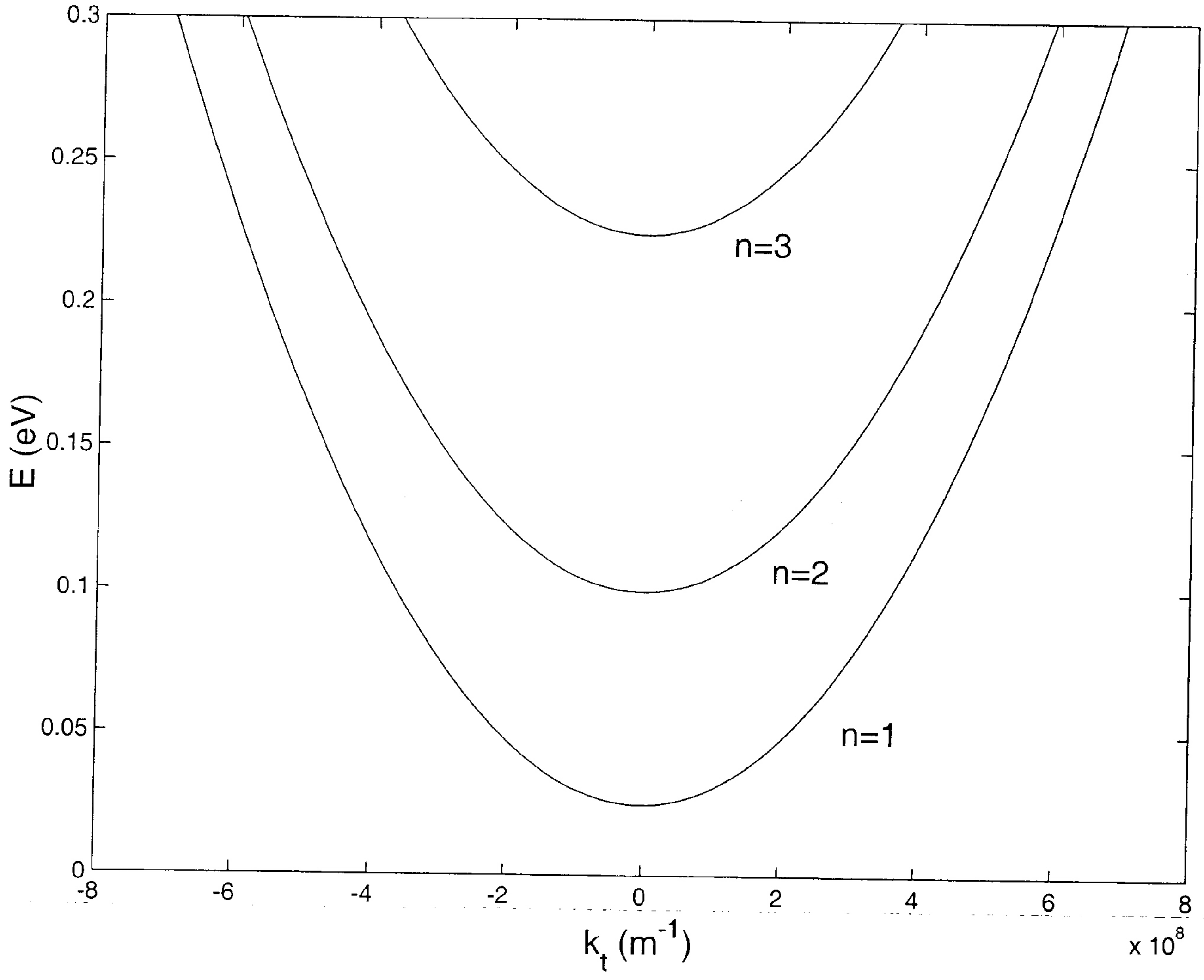
$$k_f = \sqrt{\frac{2m^*}{\hbar^2}(E - E_n)}$$

N_T = total number of states with $E < E'$

$$= \frac{2}{(2\pi)^2}\pi(k_{f1})^2 = \frac{2\pi}{(2\pi)^2}\left[\frac{2m^*}{\hbar^2}E - \left(\frac{n\pi}{d}\right)^2\right]$$

2 spin states

$$N_1(E) = \frac{dN_T}{dE} = \frac{2\pi}{(2\pi)^2}\frac{2m^*}{\hbar^2} \quad \text{for } E > E_1$$


0.3
0.25
0.2
0.15
0.1
0.05
0
E (eV)
-8
-6
-4
-2
0
2
4
6
8
x 10^8
k_t (m^-1)
n=3
n=2
n=1

Density of states (n=2)

$$N_2(E) = 0 \quad \text{for } E < E_2 \qquad \left(E_2 = \frac{\hbar^2}{2m^*}\left(\frac{2\pi}{d}\right)^2\right)$$

$$N_T = \frac{2}{(2\pi)^2}\pi\left(k_{f2}\right)^2 = \frac{2\pi}{(2\pi)^2}\left[\frac{2m^*}{\hbar^2}E - \left(\frac{2\pi}{d}\right)^2\right]$$

$$N_2(E) = \frac{dN_T}{dE} = \frac{2\pi}{(2\pi)^2}\frac{2m^*}{\hbar^2} \quad \text{for } E > E_2$$

Similarly: $N_3(E) = \frac{2\pi}{(2\pi)^2}\frac{2m^*}{\hbar^2}$ for $E > E_3$

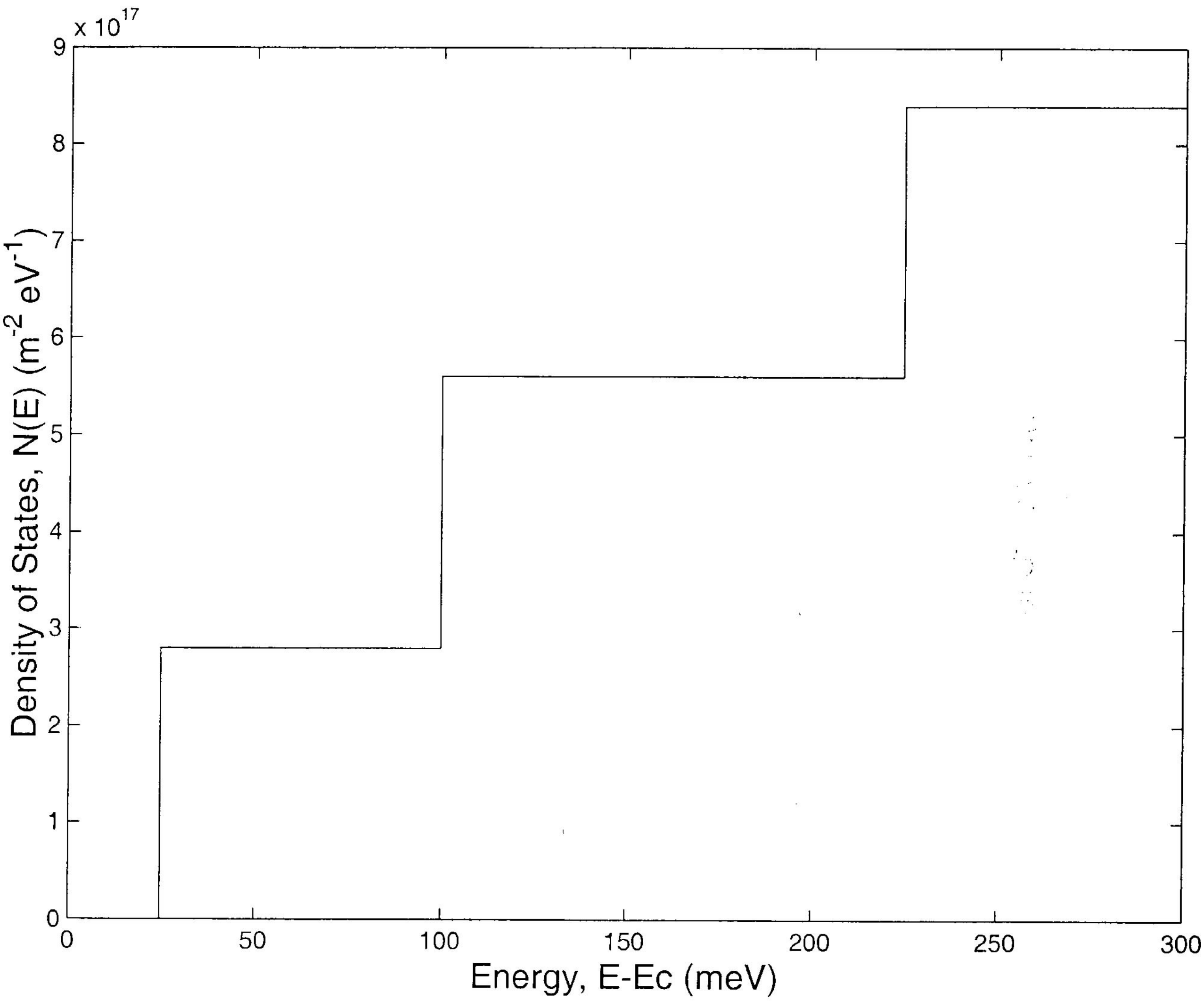
$$N(E) = N_1 + N_2 + N_3 + \ldots$$

Units: $N(E) \longleftrightarrow \frac{(kg)}{(J\text{-}sec)^2} = \frac{(kg)}{J^2\,sec^2} = \frac{(kg)}{kg\,m^2/sec^2\;sec^2\,J} = m^{-2}J^{-1}$

= electron states/m^2/J

which is correct for 2D density of states

(To convert to electron states/m^2/eV, multiply by 1.6×10^{-19})


x 10^17
Density of States, N(E) (m^-2 eV^-1)
Energy, E-Ec (meV)
0
50
100
150
200
250
300
0
1
2
3
4
5
6
7
8
9